Development of an Unconditionally Stable Full-wave 2D ADI-FDTD Method for Analysis of Arbitrary Wave Guiding Structures

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***Abstract* — This paper presents the development of unconditionally stable full-wave 2D ADI-FDTD method for analysis of characteristics of arbitrary uniform wave guiding structures. The method is derived by assuming the field variation of $e^{-j\beta z}$ along the z-direction and multiplying the field equations with an additional factor j in the recently developed 3-dimensional ADI-FDTD algorithm. In difference from the conventional full-wave 2D FDTD, it does not require that the time step be bounded by the stability condition. As a result, much of CPU time and memory can be saved. The dispersion relation of the method is also presented and is used to determine effects of discretization parameters on the accuracy. To validate the method, a boxed microstrip line on an anisotropic sapphire substrate is calculated and the results are compared with those obtained with other methods.**

I. Introduction

Recently a 3D alternating-direction-implicit (ADI) method was developed and applied to the solutions of Maxwell's equations using a variation of FDTD method [1][2]. This ADI-FDTD method eliminates the CFL stability condition and is unconditionally stable. As a result, in the regions of a solution domain that require variable meshes, the time step can be taken uniformly the same as that used in the coarse grid. That leads to reductions in potentially significant computational expenditures, such as CPU time.

Although the ADI-FDTD is more efficient than the conventional FDTD in certain cases [2], it suffers the disadvantage of inefficiency and inaccuracy when it is applied for the full-wave analysis of modes in longitudinally uniform wave guiding structures. The reason is that with the 3D ADI-FDTD method, either time and memory consuming three dimensional simulations are required or stable super-absorbing boundary conditions are needed(which are hard to find in particular for the inhomogeneous guides). With the conventional 3D FDTD, the same problems exist. As a result, the compact 2D FDTD method was developed [3][4]. However, the compact 2D method is conditionally stable and the CFL stability condition still remains [5]. The time step has to be small when the spatial discretization step is small. For a non-uniform mesh, the time step may become very small due to the small fine-mesh size. That may lead to requirement of a large number of iterations and therefore large CPU time.

In this paper, we apply the principle of the compact 2D technique to the 3D ADI-FDTD method and develop a novel unconditionally stable full-wave 2D ADI-FDTD method for analysis of wave guiding structures. In comparisons with the 3D ADI-FDTD approach, the proposed scheme presents much improved efficiency and accuracy. In comparisons with the conventional compact 2D-FDTD techniques, the proposed scheme allows a larger time step to be used irrespective of the spatial step and propagation constant, leading to improvements in computation efficiency in particular in a non-uniform mesh setting.

II. FORMULATIONS

The full-wave 2D ADI-FDTD is derived from the three dimensional ADI-FDTD [1]. By applying the procedure described in [4] to the 3D ADI-FDTD schemes [1] and assuming

$$E_x^n(x,y,z), E_y^n(x,y,z), H_z^x(x,y,z) = \{E_x^n(x,y), E_y^n(x,y), H_z^x(x,y)\} j\exp\{-j\beta z\} \qquad (1)$$

$$H_x^n(x,y,z), H_y^n(x,y,z), E_z^x(x,y,z) = \{H_x^n(x,y), H_y^n(x,y), E_z^x(x,y)\} \exp\{-j\beta z\} \qquad (2)$$

We can obtain the following unconditionally stable full-wave 2D ADI-FDTD formulations:

1) at (n+1/2)th time step,

$$E^{n+1/2}_{x(i+\frac{1}{2},j)} = E^{n}_{x(i+\frac{1}{2},j)} + \frac{\Delta t}{2\varepsilon}[(H^{n+1/2}_{z(i+\frac{1}{2},j+\frac{1}{2})} - H^{n+1/2}_{z(i+\frac{1}{2},j-\frac{1}{2})})/\Delta y + \beta H^{n}_{y(i+\frac{1}{2},j)}] \qquad (3)$$

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2) at (n+1)th time step,

$$E^{n+1}_{x(i+\frac{1}{2},j)} = E^{n+1/2}_{x(i+\frac{1}{2},j)} + \frac{\Delta t}{2\varepsilon}[(H^{n+1/2}_{z(i+\frac{1}{2},j+\frac{1}{2})} - H^{n+1/2}_{z(i+\frac{1}{2},j-\frac{1}{2})})/\Delta y + \beta H^{n+1}_{y(i+\frac{1}{2},j)}] \quad (4)$$

Equations for other field components can be obtained similarly. The further simplification of the above equations to the wave-equation-like equations for computation can also be made in a way similar to that described in [1].

It can be seen that there are no third dimensional discretization step Δz and the associated index k any more. Therefore this method reduces a 3D ADI-FDTD analysis to a two dimensional analysis.

III. STABILITY AND DISPERSION ANALYSIS

The proof of the unconditional stability of full-wave 2D ADI-FDTD follows the analysis presented in [1] for the 3D ADI-FDTD method in a homogeneous, lossless medium.

By assuming the spatial frequencies to be k_x and k_y along the x and y directions, the field components in the spatial spectral domain can be written as

$$E^n_{\alpha(m,n)} = E^n_\alpha e^{-j(k_x m\Delta x + k_y n\Delta y)}$$
$$H^n_{\alpha(m,n)} = H^n_\alpha e^{-j(k_x m\Delta x + k_y n\Delta y)} \quad \alpha = x,y,z \quad (5)$$

Substitution of these equations into (3), (4) and the like equations for other field components can lead to

$$X^{n+1} = \Lambda_2 X^{n+1/2} = \Lambda_2\Lambda_1 X^n = \Lambda X^n \quad (6)$$

with

$$X^n = [E^n_x, E^n_y, E^n_z, H^n_x, H^n_y, H^n_z]^T$$

$$\Lambda_1 = \begin{bmatrix} \frac{1}{Q_y} & \frac{W_xW_y}{Q_y} & 0 & 0 & \frac{\eta W_z}{Q_y} & \frac{-j\eta W_y}{Q_y} \\ 0 & \frac{1}{Q_z} & \frac{-jW_zW_y}{Q_z} & \frac{-\eta W_z}{Q_z} & 0 & \frac{j\eta W_x}{Q_z} \\ \frac{jW_zW_x}{Q_x} & 0 & \frac{1}{Q_x} & \frac{j\eta W_y}{Q_x} & \frac{-j\eta W_x}{Q_x} & 0 \\ 0 & \frac{W_z}{\eta Q_z} & \frac{jW_y}{\eta Q_z} & \frac{1}{Q_z} & 0 & \frac{jW_xW_y}{Q_z} \\ \frac{-W_z}{\eta Q_x} & 0 & \frac{-jW_x}{\eta Q_x} & \frac{W_xW_y}{Q_x} & \frac{1}{Q_x} & 0 \\ \frac{-jW_y}{\eta Q_y} & \frac{jW_x}{\eta Q_y} & 0 & 0 & \frac{-jW_zW_y}{Q_y} & \frac{1}{Q_y} \end{bmatrix}$$

$$\Lambda_2 = \begin{bmatrix} \frac{1}{Q_z} & 0 & \frac{-jW_xW_z}{Q_z} & 0 & \frac{\eta W_z}{Q_z} & \frac{-j\eta W_y}{Q_z} \\ \frac{W_xW_y}{Q_x} & \frac{1}{Q_x} & 0 & \frac{-\eta W_z}{Q_x} & 0 & \frac{j\eta W_x}{Q_x} \\ 0 & \frac{jW_zW_y}{Q_y} & \frac{1}{Q_y} & \frac{j\eta W_y}{Q_y} & \frac{-j\eta W_x}{Q_y} & 0 \\ 0 & \frac{W_z}{\eta Q_y} & \frac{jW_y}{\eta Q_y} & \frac{1}{Q_y} & \frac{W_xW_y}{Q_y} & 0 \\ \frac{-W_z}{\eta Q_z} & 0 & \frac{-jW_x}{\eta Q_z} & 0 & \frac{1}{Q_z} & \frac{jW_zW_y}{Q_z} \\ \frac{-jW_y}{\eta Q_x} & \frac{jW_x}{\eta Q_x} & 0 & \frac{-jW_xW_z}{Q_x} & 0 & \frac{1}{Q_x} \end{bmatrix}$$

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \quad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$W_x = \frac{v\Delta t}{\Delta x}\sin(\frac{k_x\Delta x}{2}) \quad W_y = \frac{v\Delta t}{\Delta y}\sin(\frac{k_y\Delta y}{2}) \quad W_z = \frac{v\Delta t}{2}\beta$$

$$Q_\alpha = 1 + W_\alpha^2, \alpha = x,y,z$$

It is not difficult to prove, with the help of Maple 5.1, that all the eigenvalues of Λ have the magnitude of unity. That means the full-wave 2D ADI-FDTD retains the property of uncondional stability as its 3D ADI-FDTD counterpart.

The numerical dispersion relation for the full-wave 2D ADI-FDTD scheme can be obtained as described in [6],

$$\sin^2(\omega\Delta t) = 4\frac{(W_x^2 + W_y^2 + W_z^2 + W_x^2W_y^2 + W_x^2W_z^2 + W_y^2W_z^2)(W_x^2W_y^2W_z^2 + 1)}{(1+W_x^2)^2(1+W_y^2)^2(1+W_z^2)^2} \quad (7)$$

An air-filled $(a \times b = 9mm \times 6mm)$ metallic waveguide is used to assess the dispersion errors. For such a waveguide

$$W_x = s\sin(\frac{l\pi h}{2a}) \quad W_y = s\sin(\frac{m\pi h}{2b}) \quad W_z = \frac{s\beta h}{2} \quad (8)$$

with

$$s = \frac{v\Delta t}{h} \quad \Delta x = \Delta y = h$$

and the analytical value of the resonant wavelength for the (l, m) mode is

$$\lambda_{l,m} = \frac{2}{\sqrt{(l/a)^2 + (m/b)^2 + (\beta/\pi)^2}} \quad (9)$$

By substituting the equations (8) into (7), the numerical resonant wavelength $\lambda'_{l,m}$ is found to be:

$$\lambda'_{l,m} = \pi h s / \sin^{-1}\left\{2\left[s^2\left(s^2\beta^2h^2\sin^2(\frac{l\pi h}{2a}) + 4\sin^2(\frac{l\pi h}{2a}) + 4s^2\sin^2(\frac{l\pi h}{2a})\sin^2(\frac{m\pi h}{2b}) + 4\sin^2(\frac{m\pi h}{2b}) + \beta^2h^2 + s^2\sin^2(\frac{m\pi h}{2b})\beta^2h^2\right)\left(s^6\sin^2(\frac{l\pi h}{2a})\sin^2(\frac{m\pi h}{2b})\beta^2h^2 + 4\right) \Big/ \left((1+s^2\sin^2(\frac{l\pi h}{2a}))^2(1+s^2\sin^2(\frac{m\pi h}{2b}))^2(4+s^2\beta^2h^2)^2\right)\right]^{1/2}\right\} \quad (10)$$

Fig. 1 and Fig.2 illustrate the variation of the relative error with the cell size h and propagation constant β respectively. The error with the compact 2D-FDTD method is also plotted for reference.

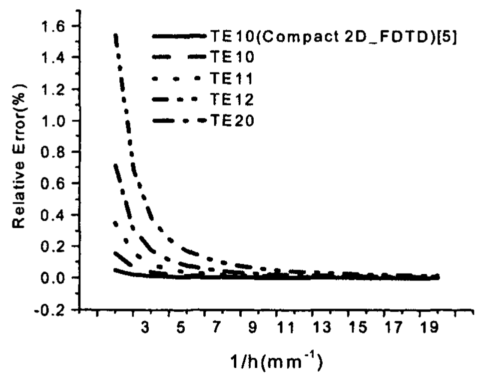

Fig. 1. Relative error versus cell size h when propagation constant β=209.4395 m^{-1} and the Courant number s=0.5.

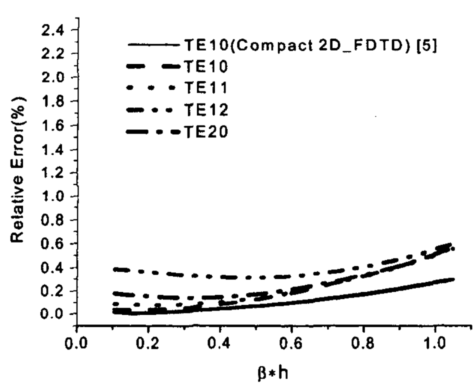

Fig. 2. Relative error versus propagation constant when h=0.5mm and Courant number s=0.5.

As can be seen, the relative error of the numerical wavelength becomes larger as h or β increase, and the relative error of the full-wave 2D ADI-FDTD and the compact 2D-FDTD are very comparable with a small h or β for lower-order modes.

Fig. 3 shows the error versus the Courant numbers, which is proportional to the time step. It suggests that the Courant number can go as high as 10 for TE10 mode with errors of less than 1.5%. This means the number of iteration with the proposed scheme can be at least 14 times fewer than the conventional 2D scheme (whose Courant number has to be less than $1/\sqrt{2}$) for TE10 mode.

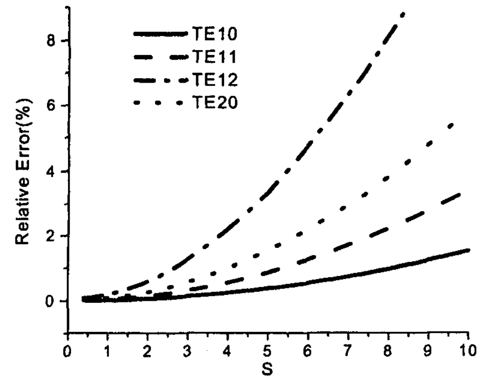

Fig. 3 Relative error versus the Courant number when propagation constant β=209.4395 m^{-1} and h=0.5mm.

IV. NUMERICAL RESULT

To validate the proposed 2D ADI-FDTD, we firstly applied it to a rectangular air-filled metallic waveguide $(a \times b = 9mm \times 6mm)$ and compared the computed dominant frequencies with analytic values and the results obtained with the 3D ADI-FDTD method using the technique described in [7]. Table I presents the comparisons. It is seen that all the errors are less than 0.2%. Furthermore, because the full-wave 2D ADI-FDTD method handles the z derivative analytically, it is more accurate than 3D ADI-FDTD when a smaller β is selected.

TABLE I
COMPARISON OF DOMINANT FREQUENCIES (GHZ)

β	Analytic Freq	ADI-FDTD		Full-wave 2D ADI-FDTD	
		Freq.	Err.%	Freq.	Err.%
104.720	17.389	17.326	0.359	17.358	0.175
209.440	19.423	19.392	0.160	19.396	0.139
314.159	22.407	22.378	0.130	22.376	0.139
418.879	26.016	25.992	0.093	25.978	0.147

The second validation example we computed is a boxed microstrip line on an anisotropic sapphire substrate, as shown in Fig 4. To facilitate the comparisons, the geometry parameters were taken the same as those in [4].

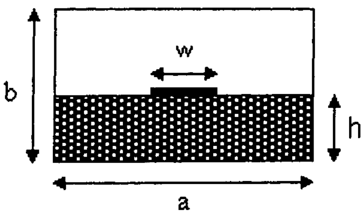

Fig. 4. boxed microstrip line on sapphire, $a = 6.5mm$ $b = 3.5mm, h = w = 1.5mm, \varepsilon_{xx} = \varepsilon_{zz} = 9.4, \varepsilon_{yy} = 11.6$

Fig.5 shows the dispersion diagram obtained by compact 2D FDTD, 3D ADI-FDTD and the proposed 2D ADI-FDTD respectively. The results are found to be in a good agreement with those methods.

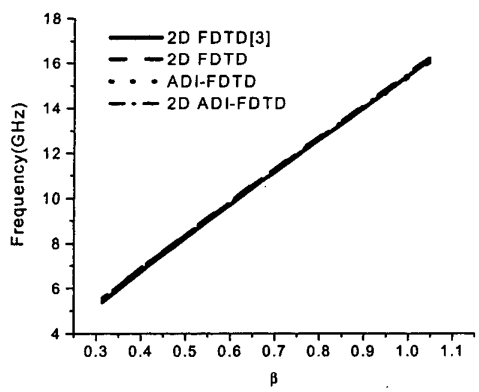

Fig. 5. Numerical comparison between Compact 2D FDTD, 3D ADI-FDTD and the proposed 2D ADI-FDTD.

These simulations were performed on AMD 950 MHz PC. The CPU time and required memory size of these simulations are shown in Table II. Note the CPU time for the compact 2D FDTD and the proposed 2D ADI-FDTD are the time for computing the five frequency points that forms the dispersion curves in Fig. 5. The CPU time for the 3D ADI-FDTD is the time for computing one of the middle frequency point where $\beta=0.6283mm^{-1}$.

TABLE II
USAGE OF CPU TIME AND MEMORY

	Δt	Steps	Time	Memory
Compact 2D FDTD	1.66ps	50000	156s	171KByte
Full-wave 2D ADI-FDTD	16.6ps	5000	70s	220KByte
3D ADI_FDTD	16.6ps	5000	2257s	11694KByte

It can be seen that with the comparable numerical accuracy, the time step of full-wave 2D ADI-FDTD can be set 5 times as large as the compact 2D FDTD. As a result, a saving factor of 2.23 in CPU time was achieved. The computation memory required with the full-wave 2D ADI-FDTD is, however, about 29% higher than that with the compact 2D FDTD method. In comparison with the 3D ADI-FDTD approach, both the CPU time and memory saving are significant. The saving factors are about 32 and 53 respectively.

V. CONCLUSION

A novel unconditionally stable 2D ADI-FDTD is developed for the analysis of hybrid modes in inhomogeneous wave guiding structures. The numerical stability and numerical dispersion of the method are presented. Through numerical experiments, it is found that the proposed 2D ADI-FDTD is generally more efficient and accurate than the 3D ADI-FDTD method and is more CPU-time efficient than compact 2D FDTD .

VI ACKNOWLEDGEMENT

The authors acknowledge the financial support for this work from Natural Science and Engineering Research Council of Canada.

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